

1. $r(t) = (-\sqrt{t}, t)$ $0 \leq t \leq 4$

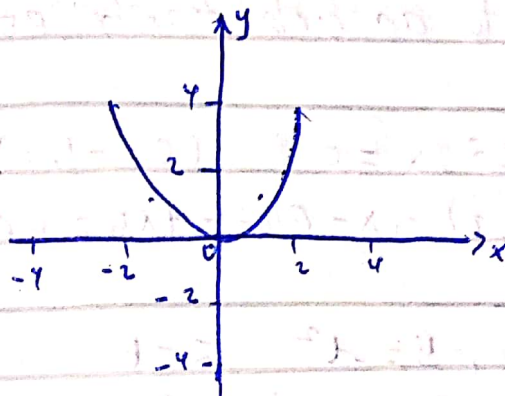
(ii) $x = -\sqrt{t}$ $x^2 = t$

$y = t$ $y = x^2$

$\Rightarrow y = x^2$

(iii)

x	y
-2	4
-1	1
0	0
1	1
2	4



(iv) Tangent vector $= r'(t) = \left(-\frac{1}{2}t^{-1/2}, 1\right)$

$\left(-\frac{1}{2}t^{-1/2}, 1\right) = \left(-\frac{1}{2\sqrt{t}}, 1\right)$

$\frac{1}{\sqrt{t}} = 4 \Rightarrow t = \frac{1}{16}$

$r'\left(\frac{1}{16}\right) = \left(-\frac{1}{2\sqrt{1/16}}, 1\right) = \left(-2, 1\right)$

(b) $f(x, y, z) = x^2 + z^2 + xy^2z$ $f(1, 3, 4) = 1 + 16 + 9(4) = 53$

(i) $f_x = 2x + y^2z$

$f_x(1, 3, 4) = 2(1) + 9(4) = 38$

$f_y = 2xyz$

$f_y(1, 3, 4) = 2(1)(3)(4) = 24$

$f_z = 2z + xy^2$

$f_z(1, 3, 4) = 2(4) + (1)(3)^2 = 17$

$f(x, y, z) = f(1, 3, 4) + (x-1)f_x(1, 3, 4) + (y-3)f_y(1, 3, 4) + (z-4)f_z(1, 3, 4)$

$= 53 + (x-1)(38) + (y-3)(24) + (z-4)(17)$

$= 38x + 24y + 17z - 125$

$f(0.9, 3.1, 3.95) = 38(0.9) + 24(3.1) + 17(3.95) - 125 = 50.75$

$$(ii) \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$u = i - 3j + 2k$$

$$\nabla f(1, 3, 4) = (38, 24, 17)$$

$$|\vec{u}| = \sqrt{(1)^2 + (-3)^2 + (2)^2} = \sqrt{14}$$

$$= \frac{\vec{u}}{|\vec{u}|} = \left\langle \frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}} \right\rangle$$

$$Df(1, 3, 4)_{\vec{u}} = (38, 24, 17) \cdot \left(\frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}} \right)$$

$$= \frac{38}{\sqrt{14}} - \frac{72}{\sqrt{14}} + \frac{34}{\sqrt{14}} = 0$$

$$2. \quad g(x, y) = -ye^{-x^2-y^2}$$

$$(i) \quad \frac{\partial g}{\partial x} = -2xy e^{-x^2-y^2}$$

$$\frac{\partial^2 g}{\partial x^2} = +2y(-2e^{-x^2-y^2} + e^{-x^2-y^2})$$

$$\frac{\partial g}{\partial y} = +2y^2 e^{-x^2-y^2} - e^{-x^2-y^2}$$

$$\frac{\partial^2 g}{\partial y^2} = -4y^3 e^{-x^2-y^2} + 6y e^{-x^2-y^2}$$

$$\frac{\partial}{\partial y} \left(\frac{\partial g}{\partial x} \right) = 2x e^{-x^2-y^2} - 4xy^2 e^{-x^2-y^2}$$

(b) Stationary points $g_x = g_y = 0$

$$-2xy e^{-x^2-y^2} = 0$$

$$(2y^2 - 1) e^{-x^2-y^2} = 0$$

$$e^{-x^2-y^2} \neq 0$$

$$e^{-x^2-y^2} \neq 0$$

$$-2xy = 0$$

$$(2y^2 - 1) = 0$$

$$x=0, y=0$$

$$2y^2 = 1$$

$$y = \pm \frac{1}{\sqrt{2}}$$

$y = \pm \frac{1}{\sqrt{2}}$ is not valid, therefore, stationary point = (0,0)

(b) A tangent vector is a vector with that is parallel to a curve at a given point.

An intrinsic equation of a curve defines the shape of a curve without specifying its position relative to an arbitrary point.

(c) $y = -z^2, x = 0$ $(0, -1, 1)$ to $(0, -4, 2)$

$$Q(x, y, z) = x - 6y^2 - 4xyz + 6z^4 - z$$

$$x = 0, y = -t^2, z = t$$

$$\frac{dx}{dt} = 0, \quad dy = -2t dt, \quad dz = dt$$

$$\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

$$= -t^2\mathbf{j} + t\mathbf{k}$$

$$\vec{r}'(t) = -2t\mathbf{j} + \mathbf{k}$$

$$|\vec{r}'(t)| = \sqrt{(-2t)^2 + 1} = \sqrt{4t^2 + 1}$$

$$\int_C Q \, ds = \int_1^2 Q(r(t)) |\vec{r}'(t)| \, dt$$

$$Q(r(t)) = 0 - 6(-t^2)^2 - 4(0)(-t^2)(t) + 6(t)^4 - t$$

$$= -6t^4 + 6t^4 - t = -t$$

$$= \int_1^2 -t \sqrt{4t^2 + 1} \, dt$$

$$u = 4t^2 + 1$$

$$du = 8t \, dt$$

$$= -\frac{1}{8} \int_1^2 \sqrt{u} \, du = -\frac{1}{8} \times \frac{2}{3} \left[u^{3/2} \right]_1^2 = -\frac{1}{12} \left[(4t^2 + 1)^{3/2} \right]_1^2$$

$$= -\frac{17\sqrt{17} - 5\sqrt{5}}{12}$$

$$3(a) \quad y = -\sqrt{x}, \quad z = x$$

$$\text{Let } x = t, \quad y = -\sqrt{t}, \quad z = t$$

$$r(t) = \langle t, -\sqrt{t}, t \rangle$$

$$r'(t) = \langle 1, -\frac{1}{2}(t)^{-1/2}, 1 \rangle$$

$$W = \int xz dx + \int -4xyz dy + \int 20z dz$$

$$= \int \left(t^2 - 4(t)(-\sqrt{t})(t) \left(-\frac{1}{2\sqrt{t}} \right) + 20t \right) dt$$

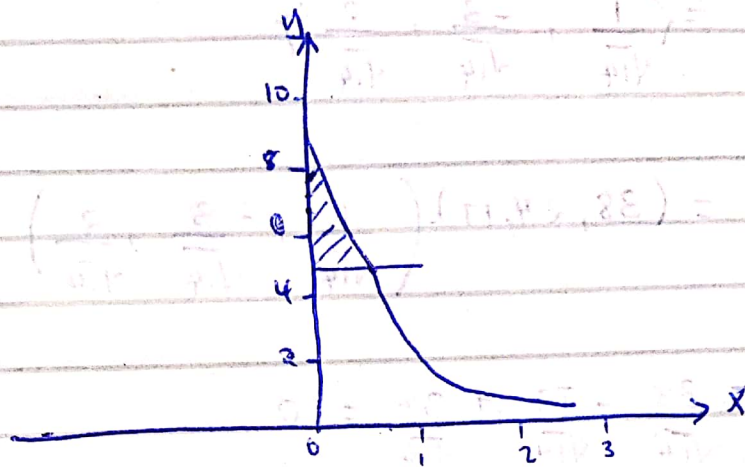
$$= \int_4^5 (t^2 - 2t^2 + 20t) dt$$

$$= \int_4^5 (-t^2 + 20t) dt = -\frac{t^3}{3} + 10t^2 \bigg|_4^5$$

$$= \left(-\frac{125}{3} + 250 \right) - \left(-\frac{64}{3} + 160 \right)$$

$$= \frac{209}{3}$$

(b) (i)



(ii) $5 \leq y \leq 9e^{-x}$
 $0 \leq x$

$$y = 9e^{-x}$$

$$e^{-x} = \frac{y}{9} \quad x = -\ln\left(\frac{y}{9}\right)$$

$$= \int_5^9 -\ln\left(\frac{y}{9}\right) dy$$

(iii) $\int_5^9 -\ln\left(\frac{y}{9}\right) dy$

$$u = \ln\left(\frac{y}{9}\right) \quad du = \frac{1}{y} dy$$

$$dv = 1 dy \quad v = y$$

$$= y \ln\left(\frac{y}{9}\right) - \int 1 dy = \left[y \ln\left(\frac{y}{9}\right) - y \right]_5^9$$

$$= (9 \ln(1) - 9) - (5 \ln\left(\frac{5}{9}\right) - 5)$$

$$= 5 \ln\left(\frac{5}{9}\right) + 4$$

$$(C) \quad y^2 + z^2 = 25, \quad x = 0 \quad z^2 = 25 - y^2$$

$$r = 5 \quad dA = r dr d\theta$$

$$A = \int_0^{2\pi} \int_0^5 r dr d\theta$$

$$= \int_0^{2\pi} \int_0^5 r(25 - r^2 \sin^2 \theta) dr d\theta = \int_0^{2\pi} \int_0^5 (25r - r^3 \sin^2 \theta) dr d\theta$$

$$= \int_0^{2\pi} \frac{1}{4} (-625 \sin^2 \theta + 1250) d\theta$$

$$= 625 \times \frac{1}{4} \left[\theta - \frac{1}{2} \sin 2\theta + 1250 \theta \right]_0^{2\pi}$$

$$= \frac{1875 \pi}{4}$$

4 (a) $x^2 + y^2 = 5^2$
 $0 \leq z \leq 25$

(b) $V = \iiint dx$ where $dv = dz dy dx$

$$y^2 = 25 - x^2$$

$$y = \pm \sqrt{25 - x^2}$$

$$V = \int_{-5}^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \int_0^{25} dz dy dx$$

(c) $x = r \cos \theta$, $y = r \sin \theta$, $z = z$

$x = 5 \cos \theta$, $y = 5 \sin \theta$, $z = z$

$dx dy dz = r dr d\theta dz$

$$V = \int_0^{2\pi} \int_0^5 \int_0^{25} r dz dr d\theta$$

$$= \int_0^{2\pi} \int_0^5 r [z]_0^{25} dr d\theta = \int_0^{2\pi} \int_0^5 25r dr d\theta$$

$$= \int_0^{2\pi} \left[\frac{25}{2} r^2 \right]_0^5 d\theta = \int_0^{2\pi} \frac{625}{2} d\theta$$

$$= \left[\frac{625}{2} \theta \right]_0^{2\pi} = 625\pi$$

(d) $x = r \cos \theta$, $y = r \sin \theta$, $z = z$

$$r(\theta, z) = 5 \cos \theta \hat{i} + 5 \sin \theta \hat{j} + z \hat{k}$$

$$(d) \quad |r_\theta \times r_z| = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial \theta}(5\cos\theta) & \frac{\partial}{\partial \theta}(5\sin\theta) & \frac{\partial}{\partial \theta}(z) \\ \frac{\partial}{\partial z}(5\cos\theta) & \frac{\partial}{\partial z}(5\sin\theta) & \frac{\partial}{\partial z}(z) \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ -5\sin\theta & 5\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 5\cos\theta i + 5\sin\theta j$$

$$= \sqrt{(5\cos\theta)^2 + (5\sin\theta)^2} = 5$$

(f)

$$S \cdot A = \iint |r_\theta \times r_z| dz d\theta$$

$$= \int_0^{2\pi} \int_0^{25} 5 dz d\theta = \int_0^{2\pi} \left[5z \right]_0^{25} d\theta = \int_0^{2\pi} 125 d\theta$$

$$= 125\theta \Big|_0^{2\pi} = 250\pi$$